Indices, Logarithms and Surds

Yue Kwok Choy

1. Prove that $\dfrac{1}{1+x^{a-b}+x^{a-c}}+\dfrac{1}{1+x^{b-c}+x^{b-a}}+\dfrac{1}{1+x^{c-a}+x^{c-b}}=\dfrac{1}{1+x^{a-b}}+\dfrac{1}{1+x^{b-a}}$

2. Given that $x^{\frac{1}{2}}+x^{-\frac{1}{2}}=3$, find the values of :

(a) $x^2 + x^{-2}$ **(b)** $x^{\frac{3}{2}}+x^{-\frac{3}{2}}$

3. If $7^{2n^2-n-3}=1$, find the possible values of n which satisfy the given equation.

4. Find the values of x in each of the following equations :

(a) $9^x-4\times 3^{x+1}+27=0$

(b) $4^{x-1}+3\times 2^{x+1}-64=0$

5. Solve the equations : $\begin{cases} 2^x+3^y=41 \\ 2^{x+2}+3^{y+2}=209 \end{cases}$

6. Simplify $\dfrac{(x^{\frac{1}{2}}y^{-\frac{1}{4}}+2x^{-\frac{1}{2}}y^{\frac{1}{4}}-2)(x+2x^{\frac{1}{2}}y^{\frac{1}{4}}+2y^{\frac{1}{2}})}{x^2+4y}$

7. Find two real values of k for which the equation $\log_{10}|x^2 + 2kx| = 0$ has a double root.

8. Solve the equation : $\dfrac{x\sqrt{x}}{x^{\log x}}=10^{-1}$

9. The quadratic equation $2x^2 \log a + (2x – 1) \log b = 0$, where a and b are positive constants, has non-zero equal roots. Express b in terms of a .

10. Let $\log_x ab=\log_a x+\log_b x$ and $ab \neq 1$. Show that $\log_x a\log_x b=1$.

11. Given that $\log_{12} 3=a$, express $\log_{\sqrt{3}} 8$ in terms of a .

12. If $y = a + bx^n$ is satified by the values :

x	1	2	4
y	7	10	15

shown that $n = \log_2\left(\dfrac{5}{3}\right)$ and deduce the values of a and b .

13. Let $\log_a c+\log_b c=0$. Show that $abc + 1 = ab + c$

14. From the definition of a logarithm, prove that $\log_a x = \dfrac{1}{\log_x a}$.

Hence, or otherwise, solve the equation : $\log_x 8 + \log_8 x = \dfrac{13}{6}$.

15. If $\log_a n = x$ and $\log_c n = y$ where $n \neq 1$, prove that $\dfrac{x - y}{x + y} = \dfrac{\log_b c - \log_b a}{\log_b c + \log_b a}$.

Verify this result, without using calculators , when $a = 4, b = 2, c = 8$ and $n = 4096$.

16. Prove from the definition of a logarithm that :

(a) $\dfrac{1}{\log_a(ab)} + \dfrac{1}{\log_b(ab)} = 1$,

(b) $\log_a b \log_b c \log_c a = 1$, and deduce the value of $\log_5 32 \log_4 7 \log_{49} 125$.

17. If $x = \dfrac{\sqrt{5} - \sqrt{3}}{\sqrt{5} + \sqrt{3}}$, find the value of $8x - x^2$.

18. Express with rational denominators :

(a) $\dfrac{1}{\sqrt{3} + \sqrt{2} - \sqrt{5}}$

(b) $\dfrac{1}{\sqrt{3} + \sqrt{2} + 1}$

19. Given that $\log_2 (x - 5y + 4) = 0$ and $\log_2 (x + 1) = 1 + 2 \log_2 y$, find the values of x and y .

20. Given that $\log_9 x = p$ and $\log_{\sqrt{3}} y = q$, express xy and $\dfrac{x^2}{y}$ as powers of 3.

21. Find the value of $\dfrac{a^3 + 2a^2b + b^3}{ab(a + 3b)}$ when $\dfrac{a}{b} = \sqrt{2} - 1$.

22. If a, b, x, y and z are numbers greater than 1 and $a^x = b^y = (ab)^z$, show that $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{z}$.

1. $\dfrac{1}{1+x^{a-b}+x^{a-c}}=\dfrac{x^{b+c}}{x^{b+c}+x^{c+a}+x^{a+b}}$

L.H.S. = $\dfrac{x^{b+c}}{x^{b+c}+x^{b+c}x^{a-b}+x^{b+c}x^{a-c}}+\dfrac{x^{c+a}}{x^{c+a}+x^{c+a}x^{b-c}+x^{c+a}x^{b-a}}+\dfrac{x^{a+b}}{x^{a+b}+x^{a+b}x^{c-a}+x^{a+b}x^{c-b}}$

$=\dfrac{x^{b+c}}{x^{b+c}+x^{c+a}+x^{a+b}}+\dfrac{x^{c+a}}{x^{c+a}+x^{a+b}+x^{b+c}}+\dfrac{x^{a+b}}{x^{c+a}+x^{a+b}+x^{b+c}}=\dfrac{x^{b+c}+x^{c+a}+x^{a+b}}{x^{b+c}+x^{c+a}+x^{a+b}}=1$

R.H.S. = $\dfrac{1}{1+x^{a-b}}+\dfrac{1}{1+x^{b-a}}=\dfrac{x^b}{x^b+x^a}+\dfrac{x^a}{x^a+x^b}=\dfrac{x^a+x^b}{x^a+x^b}=1$ $\therefore$ L.H.S. = R.H.S.

2. **(a)** $x^{\frac{1}{2}}+x^{-\frac{1}{2}}=3$ …. (1)

$(1)^2$, $x + 2 + x^{-1} = 9$, $x + x^{-1} = 7$ …. (2)

$(2)^2$, $x^2 + 2 + x^{-2} = 49, x^2 + x^{-2} = \underline{\underline{47}}$ …. (3)

(b) $x^{\frac{3}{2}}+x^{-\frac{3}{2}}=\left(x^{\frac{1}{2}}\right)^3+\left(x^{-\frac{1}{2}}\right)^3=\left(x^{\frac{1}{2}}+x^{-\frac{1}{2}}\right)\left(x-1+x^{-1}\right)=3(7-1)=\underline{\underline{18}}$

3. $7^{2n^2-n-3}=1$ $\Rightarrow 7^{2n^2-n-3}=7^0$ $\Rightarrow 2n^2-n-3=0$ $\Rightarrow (n+1)(2n-3)=0$

$\therefore$ $n=-1,\ \dfrac{3}{2}$

4. **(a)** $9^x-4\times3^{x+1}+27=0\Rightarrow\left(3^x\right)^2-12\left(3^x\right)+27=0\Rightarrow\left[3^x-3\right]\left[3^x-9\right]=0\Rightarrow3^x=3\ \text{or}\ 3^2$

$\therefore$ x = 1 or 2

(b) $4^{x-1}+3\times2^{x+1}-64=0\Rightarrow2^{2x-2}+6\times2^x-64=0$ $\Rightarrow\left(2^x\right)^2+24\times2^x-256=0$

$\Rightarrow\left[2^x-8\right]\left[2^x+32\right]=0\Rightarrow2^x=8=2^3,$ $2^x+32=0$ is rejected.

$\therefore$ x = 3 .

5. Put $u = 2^x$, $v = 3^y$, $\begin{cases}2^x+3^y=41\\2^{x+2}+3^{y+2}=209\end{cases}\Rightarrow\begin{cases}u+v=41\\4u+9v=209\end{cases}\Rightarrow\begin{cases}u=32\\v=9\end{cases}\Rightarrow\begin{cases}2^x=2^5\\3^y=3^2\end{cases}\Rightarrow\begin{cases}x=5\\y=2\end{cases}$

6. $E=\dfrac{(x^{\frac{1}{2}}y^{-\frac{1}{4}}+2x^{-\frac{1}{2}}y^{\frac{1}{4}}-2)(x+2x^{\frac{1}{2}}y^{\frac{1}{4}}+2y^{\frac{1}{2}})}{x^2+4y}$. Let $u=x^{\frac{1}{2}}$, $v=y^{\frac{1}{4}}$, so that $u^2 = x$, $v^4 = y$.

$E=\dfrac{(uv^{-1}+2u^{-1}v-2)(u^2+2uv+2v^2)}{u^4+4v^4}=\dfrac{(u^2+2v^2-2uv)(u^2+2uv+2v^2)}{uv\left(u^4+4v^4\right)}=\dfrac{(u^2+2v^2)^2-(2uv)^2}{uv\left(u^4+4v^4\right)}$

$=\dfrac{u^4+4v^4}{uv\left(u^4+4v^4\right)}=\dfrac{1}{uv}=x^{-\frac{1}{2}}y^{-\frac{1}{4}}$

7. $\log_{10}|x^2 + 2kx| = 0$ $\Rightarrow$ $|x^2 + 2kx| = 10^0 = 1$ $\Rightarrow$ $x^2 + 2kx = \pm1$ $\Rightarrow$ $x^2 + 2kx \pm1 = 0$

$\Rightarrow$ $x^2 + 2kx + 1 = 0$ or $x^2 + 2kx - 1 = 0$

Since the given equation has double roots $\Rightarrow$ $\Delta_1 = (2k)^2 - 4(1)(1) = 0$ or $\Delta_2 = (2k)^2 - 4(1)(-1) = 0$

$\Rightarrow$ $k = \pm1$ ($\Delta_2 = 4k^2 + 4 = 0$ has no solution)

8. $\frac{x\sqrt{x}}{x^{\log x}} = 10^{-1} \Rightarrow x^{\frac{3}{2}-\log x} = 10^{-1} \Rightarrow \log x^{\frac{3}{2}-\log x} = -1 \Rightarrow \left(\frac{3}{2} - \log x\right)\log x = -1 \Rightarrow (\log x)^2 - \frac{3}{2}\log x - 1 = 0$

$\Rightarrow 2(\log x)^2 - 3\log x - 2 = 0 \qquad \Rightarrow (2\log x + 1)(\log x - 2) = 0 \qquad \Rightarrow \log x = -\frac{1}{2}$ or $\log x = 2$

$\Rightarrow x = 10^{-\frac{1}{2}}$ or $10^2 \qquad \Rightarrow x = \frac{1}{\sqrt{10}}$ or 100

9. $2x^2 \log a + (2x - 1) \log b = 0$ or $2x^2 \log a + 2x \log b - \log b = 0$ has non-zero equal roots.

$\therefore \quad \Delta = 0$ and $2\log b \neq 0$

$\therefore \quad (2\log b)^2 - 4 (2\log a) (-\log b) = 0 \qquad \Rightarrow \quad 4\log b (\log b + 2 \log a) = 0$

Since $\log b \neq 0$, $\therefore \quad \log b + 2 \log a = 0 \qquad \Rightarrow \quad \log ba^2 = 0 \qquad \Rightarrow \quad ba^2 = 1 \qquad \Rightarrow \quad b = a^{-2}$.

10. $\log_x ab = \log_x a + \log_x b$. Let $u = \log_x a$, $\quad v = \log_x b$.

Then $\quad \log_x ab = \log_a x + \log_b x \Rightarrow u + v = \frac{1}{u} + \frac{1}{v} \Rightarrow u + v = \frac{u+v}{uv} \Rightarrow (uv - 1)(u + v) = 0$

$\Rightarrow u + v = 0 \quad$ or $\quad uv = 1$

If $u + v = 0$, then $\log_x a + \log_x b = 0$, $\log_x ab = 0$ and $ab = 1$. This contradicts to $ab \neq 1$.

$\therefore \quad uv = 1$

$\therefore \quad \log_x a \log_x b = 1$.

11. $\log_{\sqrt{3}} 8 = \frac{\log_{12} 8}{\log_{12}\sqrt{3}} = \frac{\log_{12} 2^3}{\log_{12} 3^{1/2}} = \frac{3}{1/2}\left(\frac{\log_{12} 2}{\log_{12} 3}\right) = 6\left(\frac{\log_{12} 2}{\log_{12} 3}\right) = 3\left(\frac{\log_{12} 2^2}{\log_{12} 3}\right) = 3\left(\frac{\log_{12}(12/3)}{\log_{12} 3}\right)$

$= 3\left(\frac{\log_{12} 12 - \log_{12} 3}{\log_{12} 3}\right) = 3\left(\frac{1-a}{a}\right)$

12. $7 = a + b$ …. (1) , $\quad 10 = a + b(2^n)$ …. (2) , $\quad 15 = a + b(4^n)$ …. (3)

(2) – (1), $\quad 3 = b(2^n - 1)$ …. (3)

(3) – (2), $\quad 5 = b(4^n - 2^n) = b2^n (2^n - 1)$ …. (4)

(4)/(3), $\quad \frac{5}{3} = 2^n$, $\quad \therefore \quad n = \log_2\left(\frac{5}{3}\right)$

From (3), $\quad 3 = b(5/3 - 1) \quad \therefore \quad b = 4.5$

From (1), $\quad a = 2.5$

13. $\log_a c + \log_b c = 0 \Rightarrow \frac{\log c}{\log a} + \frac{\log c}{\log b} = 0 \Rightarrow \frac{\log c(\log a + \log b)}{\log a \log b} = 0 \Rightarrow \frac{\log c(\log ab)}{\log a \log b} = 0$

$\Rightarrow \log c = 0 \quad$ or $\quad \log ab = 0 \qquad \Rightarrow c = 1$ or $\quad ab = 1$.

If $c = 1$, $\qquad$ L.H.S. = ab + c = R.H.S.

If $ab = 1$, $\qquad$ L.H.S. = c + 1 = R.H.S.

14. $y = \log_a x \quad \Rightarrow a^y = x \quad \Rightarrow a = x^{\frac{1}{y}} \quad \Rightarrow \log_x a = \frac{1}{y} \quad \Rightarrow y = \frac{1}{\log_x a}$.

$$\log_x 8 + \log_8 x = \frac{13}{6} \quad \Rightarrow \frac{1}{\log_8 x} + \log_8 x = \frac{13}{6} \Rightarrow 6(\log_8 x)^2 - 13(\log_8 x) + 6 = 0$$

$$\Rightarrow (2\log_8 x - 3)(3\log_8 x - 2) = 0 \quad \Rightarrow \log_8 x = \frac{3}{2}, \frac{2}{3} \Rightarrow x = 8^{\frac{3}{2}}, 8^{\frac{2}{3}} \Rightarrow x = \left(2^3\right)^{\frac{3}{2}}, \left(2^3\right)^{\frac{2}{3}}$$

$$\Rightarrow x = 16\sqrt{2}, 4$$

15. $\frac{x-y}{x+y} = \frac{\log_a n - \log_c n}{\log_a n + \log_c n} = \frac{\frac{\log_b n}{\log_b a} - \frac{\log_b n}{\log_b c}}{\frac{\log_b n}{\log_b a} + \frac{\log_b n}{\log_b c}} = \frac{\log_b c - \log_b a}{\log_b c + \log_b a}$.

$$\frac{x-y}{x+y} = \frac{\log_4 4096 - \log_8 4096}{\log_4 4096 + \log_8 4096} = \frac{\log_4 4^6 - \log_8 8^4}{\log_4 4^6 + \log_8 8^4} = \frac{6\log_4 4 - 4\log_8 8}{6\log_4 4 + 4\log_8 8} = \frac{6-4}{6+4} = \frac{1}{5}$$

$\frac{\log_b c - \log_b a}{\log_b c + \log_b a} = \frac{\log_2 8 - \log_2 4}{\log_2 8 + \log_2 4} = \frac{\log_2 2^3 - \log_2 2^2}{\log_2 2^3 + \log_2 2^2} = \frac{3\log_2 2 - 2\log_2 2}{3\log_2 2 + 2\log_2 2} = \frac{3-2}{3+2} = \frac{1}{5}$. Result follows.

16. **(a)** Let $x = \log_a (ab)$, $y = \log_b (ab)$. Therefore $a^x = ab$, $b^y = ab$.

$a = (ab)^{\frac{1}{x}}$, $b = (ab)^{\frac{1}{y}}$ $\Rightarrow ab = (ab)^{\frac{1}{x}}(ab)^{\frac{1}{y}} = (ab)^{\frac{1}{x}+\frac{1}{y}}$ $\Rightarrow \frac{1}{x} + \frac{1}{y} = 1$. Result follows.

(b) $x = \log_a b$, $y = \log_b c$ $z = \log_c a$ $\Rightarrow$ $a^x = b$, $b^y = c$, $c^z = a$.

Hence $a^{xyz} = b^{yz} = c^z = a$ $\Rightarrow$ $xyz = 1$. Result follows.

$\log_5 32 \log_4 7 \log_{49} 125 = \log_5 2^5 \log_4 7 \log_{49} 5^3 = (5\log_5 2)(\log_4 7)(3\log_{49} 5)$

$= 15 (\log_5 2)(\log_4 7)(\log_{49} 5) = \frac{15}{4}\left(\log_5 2^2\right)\left(\log_4 7^2\right)\left(\log_{49} 5\right) = \frac{15}{4}(\log_5 4)(\log_4 49)(\log_{49} 5)$

$= 3\frac{3}{4}$

17. $x = \frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}} = \frac{\left(\sqrt{5}-\sqrt{3}\right)^2}{\left(\sqrt{5}+\sqrt{3}\right)\left(\sqrt{5}-\sqrt{3}\right)} = \frac{5 - 2\sqrt{15} + 3}{5-3} = \frac{8 - 2\sqrt{15}}{2} = 4 - \sqrt{15}$

$\therefore$ $(x-4)^2 = 15$, $x^2 - 8x + 16 = 15$, $\therefore$ $8x - x^2 = 1$.

18. **(a)** $\frac{1}{\sqrt{3}+\sqrt{2}-\sqrt{5}} = \frac{\sqrt{3}+\sqrt{2}+\sqrt{5}}{\left(\sqrt{3}+\sqrt{2}-\sqrt{5}\right)\left(\sqrt{3}+\sqrt{2}+\sqrt{5}\right)} = \frac{\sqrt{3}+\sqrt{2}+\sqrt{5}}{\left(\sqrt{3}+\sqrt{2}\right)^2 - 5} = \frac{\sqrt{3}+\sqrt{2}+\sqrt{5}}{2\sqrt{6}}$

$= \frac{\sqrt{6}\left(\sqrt{3}+\sqrt{2}+\sqrt{5}\right)}{12}$

$= \frac{3\sqrt{2} + 2\sqrt{3} + \sqrt{30}}{12}$

(b) $\frac{1}{\sqrt{3}+\sqrt{2}+1} = \frac{1+\sqrt{2}-\sqrt{3}}{(1+\sqrt{2}+\sqrt{3})(1+\sqrt{2}-\sqrt{3})} = \frac{1+\sqrt{2}-\sqrt{3}}{(1+\sqrt{2})^2-3} = \frac{1+\sqrt{2}-\sqrt{3}}{2\sqrt{2}}$

$= \frac{\sqrt{2}(1+\sqrt{2}-\sqrt{3})}{4}$

$= \frac{2+\sqrt{2}-\sqrt{6}}{4}$

19. $\log_2(x-5y+4)=0 \quad \Rightarrow x-5y+4=2^0=1 \quad x=5y-3$ …. (1)

$\log_2(x+1)=1+2\log_2 y \quad \Rightarrow \log_2(x+1)=\log_2 2+\log_2 y^2=\log_2 2y^2 \quad \Rightarrow x+1=2y^2$ …. (2)

(1)↓(2), $5y-3+1=2y^2 \quad \Rightarrow 2y^2-5y+2=0 \quad \Rightarrow \quad y=\frac{1}{2},2$ …. (3)

(3)↓(1), $(x,y)=\left(-\frac{1}{2},\frac{1}{2}\right)$ or $(7,2)$

20. $\log_9 x = p \quad \Rightarrow \quad x = 9^p = 3^{2p}$

$\log_{\sqrt{3}} y = q \quad \Rightarrow \quad y = \sqrt{3}^p = 3^{q/2}$

$xy = 3^{2p}3^{q/2}$, $\frac{x^2}{y} = \frac{(3^{2p})^2}{3^{q/2}} = 3^{4p-\frac{q}{2}}$

21. $\frac{a^3+2a^2b+b^3}{ab(a+3b)} = \frac{\left(\frac{a}{b}\right)^3+2\left(\frac{a}{b}\right)^2+1}{\left(\frac{a}{b}\right)\left[\left(\frac{a}{b}\right)+3\right]} = \frac{(\sqrt{2}-1)^3+2(\sqrt{2}-1)^2+1}{(\sqrt{2}-1)[(\sqrt{2}-1)+3]} = \frac{(\sqrt{2}-1)(3-2\sqrt{2})+2(3-2\sqrt{2})+1}{(3-2\sqrt{2})+3(\sqrt{2}-1)}$

$= \frac{5\sqrt{2}-7+6-4\sqrt{2}+1}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2}} = 1$.

22. $a^x = b^y = (ab)^z \quad \Rightarrow \quad a = (ab)^{\frac{z}{x}}$ and $b = (ab)^{\frac{z}{y}}$

Multiply, we have, $ab = (ab)^{\frac{z}{x}}(ab)^{\frac{z}{y}} = (ab)^{\frac{z}{x}+\frac{z}{y}} \Rightarrow \frac{z}{x}+\frac{z}{y}=1 \quad \Rightarrow \quad \frac{1}{x}+\frac{1}{y}=\frac{1}{z}$.